

Transforms on operator monotone functions

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Abstract

Let f be an operator monotone function on $[0, \infty)$ with $f(t) \geq 0$ and $f(1) = 1$. If $f(t)$ is neither the constant function 1 nor the identity function t , then

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))} \quad t \geq 0$$

is also operator monotone on $[0, \infty)$, where $a, b \geq 0$ and

$$f^\sharp(t) = \frac{t}{f(t)} \quad t \geq 0.$$

Moreover, we show some extensions of this statement.

1 Introduction

We call a real continuous function $f(t)$ on an interval I operator monotone on I (in short, $f \in \mathbb{P}(I)$), if $A \leq B$ implies $f(A) \leq f(B)$ for any self-adjoint matrices A, B with their spectrum contained in I . In this paper, we consider only the case $I = [0, \infty)$ or $I = (0, \infty)$. We denote $f \in \mathbb{P}_+(I)$ if $f \in \mathbb{P}(I)$ satisfies $f(t) \geq 0$ for any $t \in I$.

Let \mathbb{H}_+ be the upper half-plane of \mathbb{C} , that is,

$$\mathbb{H}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} = \{z \in \mathbb{C} \mid |z| > 0, 0 < \arg z < \pi\},$$

where $\operatorname{Im} z$ (resp. $\arg z$) means the imaginary part (resp. the argument) of z . When we choose an element $z \in \mathbb{H}_+$, we consider that its argument satisfies $0 < \arg z < \pi$. As Loewner's theorem, it is known that f is operator monotone on I if and only if f has an analytic continuation to \mathbb{H}_+ that maps \mathbb{H}_+ into itself and also has an analytic continuation to the lower half-plane $\mathbb{H}_- (= -\mathbb{H}_+)$, obtained by the reflection across I (see [1], [3]). For an operator monotone function $f(t)$ on I , we also denote by $f(z)$ its analytic continuation to \mathbb{H}_+ .

D. Petz [5] proved that an operator monotone function $f : [0, \infty) \rightarrow [0, \infty)$ satisfying the functional equation

$$f(t) = tf(t^{-1}) \quad t \geq 0$$

is related to a Morozova-Chentsov function which gives a monotone metric on the manifold of $n \times n$ density matrices. In the work [6], the concrete functions

$$f_a(t) = a(1-a) \frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \quad (-1 < a < 2)$$

appeared and their operator monotonicity was proved (see also [2]). V.E.S. Szabo introduced an interesting idea for checking their operator monotonicity in [7]. We use a similar idea as Szabo's in our argument. M. Uchiyama [8] proved the operator monotonicity of the following extended functions:

$$\frac{(t-a)(t-b)}{(t^p - a^p)(t^{1-p} - b^{1-p})}$$

for $0 < p < 1$ and $a, b > 0$. It is well known that the function t^p ($0 \leq p \leq 1$) is operator monotone as Loewner-Heinz inequality. In this paper, we extend this statement to the following form:

Theorem 1. *Let a and b be non-negative real. If $f \in \mathbb{P}_+[0, \infty)$ and both f and f^\sharp are not constant, then*

$$h(t) = \frac{(t-a)(t-b)}{(f(t) - f(a))(f^\sharp(t) - f^\sharp(b))}$$

is operator monotone on $[0, \infty)$, where

$$f^\sharp(t) = \frac{t}{f(t)} \quad t \geq 0.$$

We can also show the operator monotonicity of other functions which have the form related to the above one in Theorem 4.

2 Main result

For $f \in \mathbb{P}[0, \infty)$, we have the following integral representation:

$$f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{z + \lambda} dw(\lambda),$$

where $\beta \geq 0$ and

$$\int_0^\infty \frac{\lambda}{1 + \lambda} dw(\lambda) < \infty$$

(see [1]). When $f(0) \geq 0$ (i.e., $f \in \mathbb{P}_+[0, \infty)$), it holds that $0 < \arg f(z) \leq \arg z$ for $z \in \mathbb{H}_+$ (i.e., $0 < \arg z < \pi$).

For any $f \in \mathbb{P}_+[0, \infty)$ ($f \neq 0$), we define f^\sharp as follows:

$$f^\sharp(t) = \frac{t}{f(t)} \quad t \in [0, \infty).$$

Then it is well-known that $f^\sharp \in \mathbb{P}_+[0, \infty)$.

Proposition 2. *Let f be an operator monotone function on $(0, \infty)$ and a be positive real.*

(1) *When $f(t)$ is not constant, the function*

$$g_1(t) = \frac{t-a}{f(t) - f(a)}.$$

is operator monotone on $[0, \infty)$.

(2) When $f(t) \geq 0$ for $t \geq 0$, the function

$$g_2(t) = \frac{f(t)(t-a)}{tf(t) - af(a)}$$

is operator monotone on $[0, \infty)$.

Proof. (1) It follows from Theorem 2.1 in [8].

(2) Since $f \in \mathbb{P}_+[0, \infty)$, we have $0 < \arg zf(z) < 2\pi$ for any $z \in \mathbb{H}_+$. So we can define

$$g_2(z) = \frac{f(z)(z-a)}{zf(z) - af(a)}, \quad z \in \mathbb{H}_+$$

and $g_2(z)$ is holomorphic on \mathbb{H}_+ . Because $g_2([0, \infty)) \subset [0, \infty)$ and $g_2(z)$ is continuous on $\mathbb{H}_+ \cup [0, \infty)$, it suffices to show that $g_2(\mathbb{H}_+) \subset \mathbb{H}_+$. By the calculation

$$\begin{aligned} g_2(z) &= \frac{zf(z) - af(a) + af(a) - f(z)a}{zf(z) - af(a)} = 1 - \frac{a(f(z) - f(a))}{zf(z) - af(a)} \\ &= 1 - \frac{a}{\frac{zf(z) - af(a)}{f(z) - f(a)}} = 1 - \frac{a}{z + f(a)g_1(z)}, \end{aligned}$$

we have

$$\operatorname{Im} g_2(z) = -\operatorname{Im} \frac{a}{z + f(a)g_1(z)} = \operatorname{Im} \frac{a(z + f(a)g_1(z))}{|z + f(a)g_1(z)|^2}.$$

When $z \in \mathbb{H}_+$, $\operatorname{Im} g_1(z) > 0$ by (1) and $\operatorname{Im} g_2(z) > 0$. So the function $g_2(t)$ belongs to $\mathbb{P}_+[0, \infty)$. \square

For any $z = e^{i\theta}$ ($0 < \theta < \pi$) and any integer $n (\geq 2)$, we set

$$w = \frac{\sin \theta}{\sin \frac{\pi + (n-1)\theta}{n}} e^{i(\pi + (n-1)\theta)/n}.$$

Since $\operatorname{Im} z = \operatorname{Im} w$, $l = z - w > 0$. Then we can get

$$\sup\{l \mid 0 < \theta < \pi\} = \lim_{\theta \rightarrow \pi-0} \frac{\sin \frac{\pi-\theta}{n}}{\sin \frac{(n-1)(\pi-\theta)}{n}} = \frac{1}{n-1}.$$

So we have the following:

Lemma 3. For any $z \in \mathbb{H}_+$ and a positive integer n ($n \geq 2$), we have

$$\arg z < \arg(z-l) < \frac{\pi + (n-1)\arg z}{n} \quad \text{if} \quad 0 < l \leq \frac{|z|}{n-1}.$$

Now we can prove the following theorem and we remark that Theorem 1 easily follows from this:

Theorem 4. Let n be a positive integer, $a, b, b_1, \dots, b_n \geq 0$ and f, g, g_1, \dots, g_n be non-constant, non-negative operator monotone functions on $[0, \infty)$.

(1) If $\frac{f(t)g(t)}{t}$ is operator monotone on $[0, \infty)$, then the function

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(g(t)-g(b))}$$

is operator monotone on $[0, \infty)$ for any $a, b \geq 0$.

(2) If $\frac{f(t)}{\prod_{i=1}^n g_i(t)}$ is operator monotone on $[0, \infty)$, then the function

$$h(t) = \frac{(t-a)}{(f(t)-f(a))} \prod_{i=1}^n \frac{g_i(t)(t-b_i)}{tg_i(t)-b_i g_i(b_i)}$$

is operator monotone on $[0, \infty)$ for any $a, b \geq 0$.

Proof. (1) By $f, g \in \mathbb{P}_+[0, \infty)$ and Proposition 2 (1),

$$\frac{t-a}{f(t)-f(a)} \text{ and } \frac{t-b}{g(t)-g(b)}$$

are operator monotone on $[0, \infty)$. Therefore

$$h(z) = \frac{(z-a)(z-b)}{(f(z)-f(a))(g(z)-g(b))}$$

is holomorphic on \mathbb{H}_+ , continuous on $\mathbb{H}_+ \cup [0, \infty)$ and satisfies $h([0, \infty)) \subset [0, \infty)$ and

$$\arg h(z) = \arg \frac{z-a}{f(z)-f(a)} + \arg \frac{z-b}{g(z)-g(b)} > 0 \text{ for } z \in \mathbb{H}_+.$$

We assume that $f(z)$ and $g(z)$ are continuous on the closure $\overline{\mathbb{H}_+}$ of \mathbb{H}_+ and

$$f(t) - f(a) \neq 0 \text{ and } g(t) - g(b) \neq 0 \text{ for any } t \in (-\infty, 0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned} \arg h(z) &= \arg(z-a) - \arg(f(z)-f(a)) + \arg(z-b) - \arg(g(z)-g(b)) \\ &\leq \pi - \arg f(z) + \pi - \arg g(z) \\ &\leq 2\pi - \arg z = \pi \quad (\text{since } \arg f(z) + \arg g(z) - \arg z \geq 0). \end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case that $z \in \mathbb{H}_+$ satisfying $|z| > \max\{a, b\}$, it holds that

$$\arg(z-a), \arg(z-b) < \frac{\pi + \arg z}{2}$$

by Lemma 3. Since

$$\begin{aligned} \arg h(z) &= \arg(z-a) - \arg(f(z)-f(a)) + \arg(z-b) - \arg(g(z)-g(b)) \\ &\leq \frac{\pi + \arg z}{2} - \arg f(z) + \frac{\pi + \arg z}{2} - \arg g(z) \\ &= \pi + \arg z - \arg f(z) - \arg g(z) \leq \pi, \end{aligned}$$

we have $0 < \arg h(z) < \pi$.

For $r > 0$, we define $H(r) = \{z \in \mathbb{C} \mid |z| \leq r, \operatorname{Im} z \geq 0\}$. Whenever $r > l = \max\{a, b\}$, we can get

$$0 \leq \arg h(z) \leq \pi$$

on the boundary of $H(r)$. Since $h(z)$ is holomorphic on $H(r)$, $\operatorname{Im} h(z)$ is harmonic on $H(r)$. Because $\operatorname{Im} h(z) \geq 0$ on the boundary of $H(r)$, we have $h(H(r)) \subset \overline{\mathbb{H}_+}$ by the minimum principle of harmonic functions. This implies

$$h(\overline{\mathbb{H}_+}) = h\left(\bigcup_{r>l} H(r)\right) \subset \bigcup_{r>l} h(H(r)) \subset \overline{\mathbb{H}_+},$$

and $h \in \mathbb{P}_+[0, \infty)$.

In general case, we set

$$\frac{f(t)g(t)}{t} = F(t) \text{ and } \tilde{f}(t) = \frac{f(t)}{F(t)} \quad (t \geq 0).$$

By the relation $\tilde{f}(t)g(t) = t$, we have $\tilde{f} \in \mathbb{P}_+[0, \infty)$. We define the function f_p , \tilde{f}_p and g_p ($0 < p < 1$) as follows:

$$f_p(z) = f(z^p), \quad \tilde{f}_p(z) = \tilde{f}(z^p),$$

and

$$g_p(z) = (\tilde{f}_p)^\sharp(z) = \frac{z}{\tilde{f}_p(z)} = \frac{zF(z^p)}{f(z^p)} = z^{1-p}g(z^p)$$

for $z \in \overline{\mathbb{H}_+}$. Then we have $f_p, g_p \in \mathbb{P}_+[0, \infty)$ and

$$h_p(z) = \frac{(z-a)(z-b)}{(f_p(z) - f_p(a))(g_p(z) - g_p(b))}$$

is holomorphic on \mathbb{H}_+ and continuous on $\overline{\mathbb{H}_+}$. By the fact $\frac{f_p(t)g_p(t)}{t} = F(t^p)$ is operator monotone on $[0, \infty)$, $h_p(t)$ becomes operator monotone on $[0, \infty)$. Since

$$\begin{aligned} h_p(t) &= \frac{(t-a)(t-b)}{(f_p(t) - f_p(a))(g_p(t) - g_p(b))} \\ &= \frac{(t-a)(t-b)}{(f(t^p) - f(a^p))(t^{1-p}g(t^p) - b^{1-p}g(b^p))} \quad \text{for } t \geq 0, \end{aligned}$$

we have

$$\lim_{p \rightarrow 1-0} h_p(t) = h(t).$$

So we can get the operator monotonicity of $h(t)$.

(2) We show this by the similar way as (1). By Proposition 2,

$$\frac{t-a}{f(t) - f(a)} \text{ and } \frac{g_i(t)(t-b_i)}{tg_i(t) - b_i g_i(b_i)} \quad (i = 1, 2, \dots, n)$$

are operator monotone on $[0, \infty)$. So we have that

$$h(z) = \frac{z-a}{f(z) - f(a)} \prod_{i=1}^n \frac{g_i(z)(z-b_i)}{zg_i(z) - b_i g_i(b_i)}$$

is holomorphic on \mathbb{H}_+ , continuous on $\mathbb{H}_+ \cup [0, \infty)$ and satisfies $h([0, \infty)) \subset [0, \infty)$ and

$$\arg h(z) = \arg \frac{z-a}{f(z)-f(a)} + \sum_{i=1}^n \arg \frac{g_i(z)(z-b_i)}{zg_i(z)-b_i g_i(b_i)} > 0$$

for $z \in \mathbb{H}_+$.

We assume that $f(z)$ and $g_i(z)$ ($i = 1, 2, \dots, n$) are continuous on $\overline{\mathbb{H}_+}$ and

$$f(t) - f(a) \neq 0 \text{ and } tg_i(t) - b_i g_i(b_i) \neq 0 \text{ for any } t \in (-\infty, 0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned} & \arg h(z) \\ &= \arg(z-a) + \sum_{i=1}^n \arg g_i(z)(z-b_i) - \arg(f(z)-f(a)) - \sum_{i=1}^n \arg(zg_i(z)-b_i g_i(b_i)) \\ &\leq \pi + \sum_{i=1}^n \arg g_i(z) + n\pi - \arg f(z) - n\pi \quad (\text{since } \arg(zg_i(z)-b_i g_i(b_i)) \geq \pi) \\ &\leq \pi \quad (\text{since } \arg f(z) - \sum_{i=1}^n \arg g_i(z) \geq 0). \end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case $z \in \mathbb{H}_+$ satisfying $|z| > n \max\{a, b_1, b_2, \dots, b_n\}$, it holds that

$$\arg(z-a), \arg(z-b) < \frac{\pi + n \arg z}{n+1}$$

by Lemma 3. We may assume that there exists a number k ($1 \leq k \leq n$) such that

$$\arg(zg_i(z)) \leq \pi \quad (i \leq k), \quad \arg(zg_i(z)) > \pi \quad (i > k).$$

Since

$$\begin{aligned} & \arg h(z) \\ &= \arg(z-a) + \sum_{i=1}^n \arg(z-b_i) + \sum_{i=1}^n \arg g_i(z) \\ &\quad - \arg(f(z)-f(a)) - \sum_{i=1}^n \arg(zg_i(z)-b_i g_i(b_i)) \\ &\leq \frac{\pi + n \arg z}{n+1} \times (n+1) + \sum_{i=1}^n \arg g_i(z) \\ &\quad - \arg f(z) - \sum_{i=1}^k \arg zg_i(z) - (n-k)\pi \\ &= \pi + n \arg z + \sum_{i=k+1}^n \arg g_i(z) - \arg f(z) - k \arg z - (n-k)\pi \\ &\leq \pi + (n-k) \arg z - (n-k)\pi \leq \pi, \end{aligned}$$

we have $0 \leq \arg h(z) \leq \pi$.

This means that it holds

$$0 \leq \arg h(z) \leq \pi$$

if z belongs to the boundary of $H(r) = \{z \in \mathbb{C} \mid |z| \leq r, \operatorname{Im} z \geq 0\}$ for a sufficiently large r . Using the same argument in (1), we can prove the operator monotonicity of h .

In general case, we define functions, for p ($0 < p < 1$), as follows:

$$f_p(t) = f(t^p), \quad g_{i,p}(t) = g_i(t^p) \quad (i = 1, 2, \dots, n).$$

Since $f, g_i \in \mathbb{P}_+[0, \infty)$,

$$0 < \arg f_p(z) < \pi, \quad 0 < \arg z g_{i,p}(z) < 2\pi$$

for $z \in \mathbb{H}_+$. This means that $f_p(z)$ and $g_{i,p}(z)$ are continuous on $\overline{\mathbb{H}_+}$ and

$$f_p(t) - f_p(a) \neq 0 \text{ and } t g_{i,p}(t) - b_i g_{i,p}(b_i) \neq 0 \text{ for any } t \in (-\infty, 0).$$

Since

$$\frac{f_p(t)}{\prod_{i=1}^n g_{i,p}(t)} = \frac{f(t^p)}{\prod_{i=1}^n g_i(t^p)} \quad (0 < p < 1)$$

is operator monotone on $[0, \infty)$, we can get the operator monotonicity of

$$\begin{aligned} h_p(t) &= \frac{t-a}{f_p(t) - f_p(a)} \prod_{i=1}^n \frac{g_{i,p}(t)(t-b_i)}{t g_{i,p}(t) - b_i g_{i,p}(b_i)} \\ &= \frac{t-a}{f(t^p) - f(a^p)} \prod_{i=1}^n \frac{g_i(t^p)(t-b_i)}{t g_i(t^p) - b_i g_i(b_i^p)}. \end{aligned}$$

So we can see that

$$h(t) = \lim_{p \rightarrow 1-0} h_p(t)$$

is operator monotone on $[0, \infty)$. □

Remark 5. Using Proposition 2 and Theorem 4, we can prove the operator monotonicity of the concrete functions in [6]. Since t^a ($0 < a < 1$) and $\log t$ is operator monotone on $(0, \infty)$,

$$\begin{aligned} f_a(t) &= a(1-a) \frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \quad (-1 < a < 2) \\ &= \begin{cases} a(a-1) \frac{t^{-a}(t-1)^2}{(t^{-a}-1)(t \cdot t^{-a}-1)} & -1 < a < 0 \\ \frac{t-1}{\log t} & a = 0, 1 \\ a(1-a) \frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} & 0 < a < 1 \\ a(a-1) \frac{t^{a-1}(t-1)^2}{(t^{a-1}-1)(t \cdot t^{a-1}-1)} & 1 < a < 2 \end{cases} \end{aligned}$$

becomes operator monotone.

Corollary 6. Let $f \in \mathbb{P}_+(0, \infty)$ and both f and f^\sharp be not constant. For any $a > 0$, we define

$$h_a(t) = \frac{(t-a)(t-a^{-1})}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1}))} \quad t \in (0, \infty).$$

Then we have

- (1) h_a is operator monotone on $(0, \infty)$.
- (2) $f(t) = t \cdot f(t^{-1})$ implies $h_a(t) = t \cdot h_a(t^{-1})$.
- (3) $a = 1$ and $f(t^{-1}) = f(t)^{-1}$ implies $h_1(t) = t \cdot h_1(t^{-1})$.

Proof. We can directly prove (1) from theorem 3. Because

$$\begin{aligned} t \cdot h_a(t^{-1}) &= \frac{t(t^{-1}-a)(t^{-1}-a^{-1})}{(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))} \\ &= \frac{(t-a)(t-a^{-1})}{t(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))}, \end{aligned}$$

we can compute

$$\begin{aligned} &t(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1})) - (f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1})) \\ &= (f(t^{-1})-f(a))(1/f(t^{-1})-t/af(a^{-1})) - (f(t)-f(a))(t/f(t)-1/af(a^{-1})) \\ &= 0 \end{aligned}$$

if it holds $f(t) = t \cdot f(t^{-1})$ or $a = 1$, $f(t^{-1}) = f(t)^{-1}$. So we have (2) and (3). \square

Example 7. Using this corollary, we can repeatedly construct an operator monotone function $h(t)$ on $[0, \infty)$ satisfying the relation

$$h(t) = t \cdot h(t^{-1}) \quad t > 0. \quad (*)$$

If we choose t^p ($0 < p < 1$) as $f(t)$ in Corollary 5(3),

$$h(t) = \frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}.$$

If we choose $\frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}$ as $f(t)$ in Corollary 5(2),

$$\begin{aligned} h(t) &= \frac{t-a}{\frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)} - \frac{(a-1)^2}{(a^p-1)(a^{1-p}-1)}} \\ &\quad \times \frac{t-a^{-1}}{\frac{t(t^p-1)(t^{1-p}-1)}{(t-1)^2} - \frac{a(a^{-p}-1)(a^{p-1}-1)}{(a-1)^2}} \end{aligned}$$

for $a > 0$. If we choose $t^p + t^{1-p}$ ($0 < p < 1$) as $f(t)$ in Corollary 5(2),

$$\begin{aligned} h(t) &= \frac{t - a}{t^p + t^{1-p} - a^p - a^{1-p}} \times \frac{t - a^{-1}}{\frac{1}{t^{p-1} + t^{-p}} - \frac{1}{a^p + a^{1-p}}} \quad (a > 0) \\ &= \frac{\sqrt{t}(\cosh(\log t) - \cosh(\log a))}{\cosh(\log \sqrt{t}) - \cosh(\log \sqrt{t} + \log(t^p + t^{1-p}) - \log(a^p + a^{1-p}))}. \end{aligned}$$

These functions, $h \in \mathbb{P}_+[0, \infty)$, satisfy the relation (*).

3 Extension of Theorem 4

Let m and n be positive integers and $f_1, f_2, \dots, f_m, g_1, g_2, \dots, g_n$ be non-constant, non-negative operator monotone functions on $[0, \infty)$. We assume that the function

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{t^{m-1} \prod_{j=1}^n g_j(t)}$$

is operator monotone on $[0, \infty)$. For non-negative numbers $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$, we define the function $h(t)$ as follows:

$$h(t) = \prod_{i=1}^m \frac{t - a_i}{f_i(t) - f_i(a_i)} \prod_{j=1}^n \frac{g_j(t)(t - b_j)}{tg_j(t) - b_jg_j(b_j)} \quad (t \geq 0).$$

Then it follows from Proposition 2 that $h(z)$ is holomorphic on $\mathbb{H}_+, h([0, \infty)) \subset [0, \infty)$ and $\arg h(z) > 0$ for any $z \in \mathbb{H}_+$.

Theorem 8. In the above setting, we have the followings:

(1) When f_i and g_j ($1 \leq i \leq m, 1 \leq j \leq n$) are continuous on $\overline{\mathbb{H}_+}$ and

$$f_i(t) - f_i(a_i) \neq 0, \quad tg_j(t) - b_jg_j(b_j) \neq 0, \quad t \in (-\infty, 0),$$

$h(t)$ is operator monotone on $[0, \infty)$.

(2) When there exists a positive number α such that $\alpha \arg z \leq \arg F(z)$ for all $z \in \mathbb{H}_+$, $h(t)$ is operator monotone on $[0, \infty)$.

Proof. (1) Using the same argument of proof of Theorem 4 (1), it suffices to show that $0 \leq \arg h(z) \leq \pi$ for $z \in \mathbb{R}$ or $z \in \mathbb{H}_+$ whose absolute value is sufficiently large.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned} &\arg h(z) \\ &= \sum_{i=1}^m \arg(z - a_i) + \sum_{j=1}^n \arg(g_j(z)(z - b_j)) \\ &\quad - \sum_{i=1}^m \arg(f_i(z) - f_i(a_i)) - \sum_{j=1}^n \arg(zg_j(z) - b_jg_j(b_j)) \\ &\leq m\pi + n\pi + \sum_{j=1}^n \arg g_j(z) - \sum_{i=1}^m \arg f_i(z) - n\pi \\ &= \pi - \arg \frac{\prod_{i=1}^m f_i(z)}{z^{m-1} \prod_{j=1}^n g_j(z)} \leq \pi. \end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case that $z \in \mathbb{H}_+$ satisfies

$$|z| > (m+n-1) \max\{a_i, b_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then it holds that

$$\arg(z - a_i), \arg(z - b_j) < \frac{\pi + (m+n-1)\arg z}{m+n}$$

by Lemma 3. We may assume that there exists k ($1 \leq k \leq n$) such that

$$\arg(zg_j(z)) \leq \pi \quad (j \leq k), \quad \arg(zg_j(z)) > \pi \quad (j > k).$$

Since

$$\begin{aligned} & \arg h(z) \\ & \leq \frac{\pi + (m+n-1)\arg z}{m+n} \times m + \frac{\pi + (m+n-1)\arg z}{m+n} \times n + \sum_{j=1}^n \arg g_j(z) \\ & \quad - \sum_{i=1}^m \arg f_i(z) - \sum_{j=1}^k \arg zg_j(z) - (n-k)\pi \\ & = \pi + (m+n-k-1)\arg z + \sum_{j=k+1}^n \arg g_j(z) - \sum_{i=1}^m \arg f_i(z) - (n-k)\pi \\ & \leq \pi + (n-k)(\arg z - \pi) - \arg \frac{\prod_{i=1}^m f_i(z)}{z^{m-1} \prod_{j=1}^m g_j(z)} \\ & \leq \pi - \arg F(z) \leq \pi, \end{aligned}$$

we have $0 \leq \arg h(z) \leq \pi$. So $h(t)$ is operator monotone on $[0, \infty)$.

(2) We choose a positive number p as follows:

$$\frac{m-1}{\alpha+m-1} < p < 1.$$

We define functions $f_{i,p}, g_{j,p}$ as follows:

$$f_{i,p}(z) = f_i(z^p), \quad g_{j,p}(z) = g_j(z^p) \quad (z \in \mathbb{H}_+).$$

Since $f_i, g_j \in \mathbb{P}_+[0, \infty)$, $f_{i,p}, g_{j,p}$ are continuous on $\overline{\mathbb{H}_+}$ and satisfy the condition

$$f_{i,p}(t) - f_{i,p}(a_i) \neq 0, \quad tg_{j,p}(t) - b_j g_{j,p}(b_j) \neq 0, \quad t \in (-\infty, 0).$$

We put

$$F_p(t) = \frac{\prod_{i=1}^m f_{i,p}(t)}{t^{m-1} \prod_{j=1}^n g_{j,p}(t)} = F(t^p)t^{-(m-1)(1-p)}.$$

Then F_p is holomorphic on \mathbb{H}_+ and satisfies $F_p((0, \infty)) \subset (0, \infty)$. For any $z \in \mathbb{H}_+$, we have

$$\arg F_p(z) = \arg F(z^p) - (m-1)(1-p)\arg z \leq \arg F(z^p) \leq \pi$$

and

$$\begin{aligned}\arg F_p(z) &\geq \alpha \arg z^p - (m-1)(1-p) \arg z \\ &= (\alpha p - (m-1)(1-p)) \arg z \\ &= ((\alpha + m-1)p - (m-1)) \arg z > 0.\end{aligned}$$

So we can see $F_p \in \mathbb{P}_+[0, \infty)$. By (1), we can show that

$$h_p(t) = \prod_{i=1}^m \frac{(t - a_i)}{f_{i,p}(t) - f_{i,p}(a_i)} \prod_{j=1}^n \frac{g_{j,p}(t)(t - b_j)}{tg_{j,p}(t) - b_j g_{j,p}(b_j)}$$

is operator monotone on $[0, \infty)$. When p tends to 1, $h_p(t)$ also tends to $h(t)$. Hence $h(t)$ is operator monotone on $[0, \infty)$. \square

Example 9. Let $0 < p_i \leq 1$ ($i = 1, 2, \dots, m$) and $0 \leq q_j \leq 1$ ($j = 1, 2, \dots, n$). We put

$$f_i(t) = t^{p_i}, \quad g_j(t) = t^{q_j} \quad (t \geq 0).$$

By the calculation

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{t^{m-1} \prod_{j=1}^n g_j(t)} = t^{\sum_{i=1}^m p_i - \sum_{j=1}^n q_j - (m-1)},$$

we have, for real numbers $a_i, b_j \geq 0$,

$$h(t) = t^{\sum_{j=1}^n q_j} \frac{(t - a_1) \cdots (t - a_m)(t - b_1) \cdots (t - b_n)}{(t^{p_1} - a_1^{p_1}) \cdots (t^{p_m} - a_m^{p_m})(t^{1+q_1} - b_1^{1+q_1}) \cdots (t^{1+q_n} - b_n^{1+q_n})}$$

is operator monotone on $[0, \infty)$ if it holds

$$0 \leq \sum_{i=1}^m p_i - \sum_{j=1}^n q_j - (m-1) \leq 1,$$

i.e., $F(t)$ is operator monotone on $[0, \infty)$.

When $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j + (m-1)$, we can see that

$$h(t) = \frac{t^{\sum_{j=1}^n q_j} (t-1)^{m+n}}{\prod_{i=1}^m (t^{p_i} - 1) \prod_{j=1}^n (t^{1+q_j} - 1)}$$

is operator monotone on $[0, \infty)$ and satisfies the functional equation

$$h(t) = t \cdot h(t^{-1}). \quad (*)$$

We can easily check that, if $h_1, h_2 \in \mathbb{P}_+[0, \infty)$ satisfy the property (*), then the functions

$$\begin{aligned}f(t) &= h_1(t)^{1/p} h_2(t)^{1-1/p} \quad (0 < p < 1) \\ g(t) &= \frac{t}{h_1(t)}\end{aligned}$$

are operator monotone on $[0, \infty)$ and satisfy the property (*).

Combining these facts, for r_i, s_i ($i = 1, 2, \dots, n$) with

$$\begin{aligned} 0 < r_1, \dots, r_c \leq 1, \quad 1 \leq r_{c+1}, \dots, r_n \leq 2 \\ 0 < s_1, \dots, s_d \leq 1, \quad 1 \leq s_{d+1}, \dots, s_n \leq 2 \\ \sum_{i=1}^c r_i = \sum_{i=c+1}^n r_i - 1, \quad \sum_{i=1}^d s_i = \sum_{i=d+1}^n s_i - 1, \end{aligned}$$

we can see that the function

$$h(t) = \sqrt{t^\gamma \prod_{i=1}^n \frac{r_i(t^{s_i} - 1)}{s_i(t^{r_i} - 1)}}$$

is operator monotone on $[0, \infty)$ and satisfies the property (*) and $h(1) = 1$, where $\gamma = 1 - c + d + \sum_{i=1}^c r_i - \sum_{i=1}^d s_i$.

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